Area Moment of Inertia
Background Principles and Motivation

Centroid

Recall the formula for a centroid:

$$\bar{x} = \frac{\int x\,dA}{\int dA}$$

The numerator of this formula is the first moment of the area:

$$Q_y = \int x\,dA$$

Moment of Inertia

Where there is a first moment, there is probably a second moment. The second moment of the area is called the moment of inertia.

$$I_x = \int y^2 dA \qquad I_y = \int x^2 dA$$

(Note: people use the term “moment of inertia” to refer to either the “area moment of inertia” or the “mass moment of inertia.” Here we're talking about the area moment of inertia.)

The moment of inertia is used among other things to measure resistance to bending in a beam.

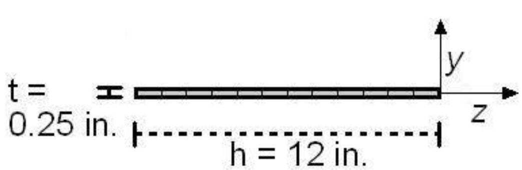
How do you calculate area moments of inertia? Just as you would calculate the integral for centroids: find the dA and take the integral. We'll also calculate area moments of inertia by composite body methods, again like is done for centroids.

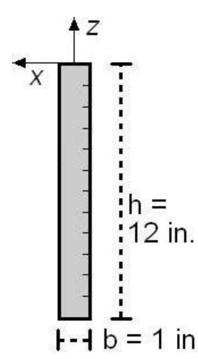
Write these two integrals down in your own hand. Try to keep them in mind as you read the rest of the page. Principally this is all we're doing on this whole page.

How does the moment of inertia relates to beam bending? After all, telling you to calculate an integral is like calculus class. The engineering part comes in the application of the integral. "What is it anyway and why do I care?"

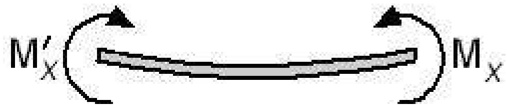
Beam Bending -- Preview for Solid Mechanics / Strength of Materials

You'll need to be able to calculate a moment of inertia to address the question of why something breaks. Think: why is it easier to bend a ruler flatwise rather than width-wise?

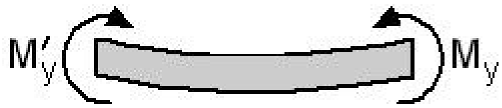




If we hold the ruler flat, and bend it about the x-axis, it bends rather easily.

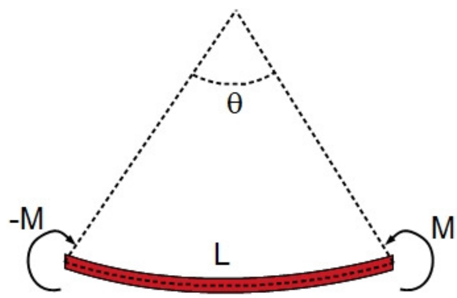


If we hold the ruler (looking down on the top) and try to bend it in the y-direction, it's much harder to get any significant movement.



All this you know, how much harder is it? To answer this question we need the moment of inertia.

Consider the springy-ness of a beam. As you apply a moment at the ends of a beam, it will bend into an arc of a circle of some radius θ.



In Solid Mechanics (or Strength of Materials), you'll prove that the moment needed to bend a beam into an arc as shown is given by

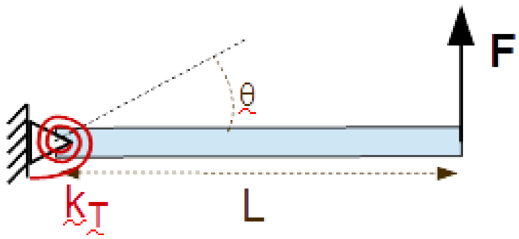
$$M = \theta\left(\frac{EI}{L}\right)$$

where

- E is a material property. (In this case, E is Young's Modulus. As the material gets harder to bend, E gets bigger. It takes a bigger moment to bend steel than wood.)
- L is the length of the beam. (It takes a bigger moment to bend a short stick than a long one.)
- and I is the area moment of inertia of the cross-section of the beam about the axis sticking out of the page. (It takes a bigger moment to bend a beam with a bigger I.)

Since the moment it takes has I in the numerator, as I gets bigger it takes more M to bend into the same arc θ. In many or even most cases, we're looking to build with beams with a higher resistance to bending, that is a beam with a higher I.

Torsional Spring – optional



You can also think of the beam's springy-ness (resistance to bending) as a torsional spring or a leaf spring.

The moment caused by a torsional spring:

$$M = k_T \Delta\theta$$

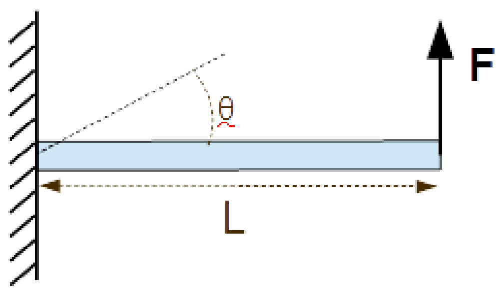
(Think of this as you would think of a linear spring where $F = k\Delta x$. A torsional spring resists an angular deformation just like a linear spring that you're more familiar with would resist an axial deformation.)

If you consider the sum of the moments on the bar above taken at the pin at the left, you'll get

$$F\ell\cos\theta = k_T \Delta\theta$$

As the beam wants to rotate about the pin, the torsional spring applies a moment equal to the spring constant times the change in the angle.

Now think of a the same beam stuck into the wall instead of held with a pin and a torsional spring.



As the force is applied, the beam will bend . The springy-ness of the beam acts like the torsional spring above: k_T = EI/L where E is the Young's modulus (the stiffness of the material), I is the moment of inertia of the cross-section, and L is the length of the beam.

Just like with the ruler, how much your beam bends up depends whether your ruler is held flat-wise or width-wise. The ruler doesn't change. What changes is the orientation of the beam. It's the cross-section of the beam that matters.